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Two-Dimensional Linear Transient Inverse Heat Conduction Problem: Boundary Condition Identification

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This article deals with the identification of unknown time- and space-dependent boundary conditions for systems driven by the heat equation. We first consider a one-dimensional and single-input problem, dealing with the identification of the time-dependent heat flux on one side of a one-dimensional linear thermal wall, from temperature and heat flux measurements on the other side. We then focus on a quenching process; our interest is to identify the time- and space-dependent heat flux on the boundary of a metal piece from temperature measurements performed inside the material (two-dimensional geometry). Those inverse problems are solved by use of a regularization method, and the solution is obtained by minimization of a quadratic criterion. Because of the linearity of the input-output relationship, the solution of this minimization is derived from the linear quadratic optimal control theory (resolution of a nonstationary Riccati equation). The robustness of the method for very small signal-to-noise ratio is shown. In the two-dimensional multi-input problem, the identification sensitivity to the localization of the measurement points is analyzed. In both cases, we consider input that are discontinuous in time, in order to show the method accuracy in the high-frequency domain.

Nomenclature

A = state equation matrix
a = thermal diffusivity
B = input matrix
C = output matrix
ERu = input error
ERy = output error
Id = identity matrix
J = optimization criterion
l = thickness of the thermal wall (one-dimensional geometry)
MU = regularization matrix (two-dimensional geometry)
n = number of spatial nodes
P = Riccati matrix
r = dimensionless radius (two-dimensional geometry)
S = step function
T = temperature
T_i = temperature at node i
t = dimensionless time variable
$\mathbf{Ucl}$ = boundary conditions vector (two-dimensional geometry)
u = input: heat flux on face $x = 1$ (one-dimensional geometry)
U_i = ith input (two-dimensional geometry)
u_0, U_{0i} = constant reference values of u in criterion J
x = dimensionless space variable (one-dimensional geometry)
Y_j = jth output (two-dimensional geometry)
y = output: temperature on face $x = 0$ (one-dimensional geometry)
z = dimensionless height (two-dimensional geometry)
β = regularization parameter (one-dimensional geometry)
Γ = time horizon
δt = time step
δx = space step
σ = standard deviation of measurement noise

Subscripts
m = measured variables
r = r direction
V = "true" variables
z = z direction

Superscripts
T = transposed vector or matrix
$*$ = identified variables

I. Introduction

THIS article deals with the identification of unknown time- and space-dependent boundary conditions for systems driven by the heat equation. Identification of boundary conditions is at stake when direct measurements on the considered boundary are unreliable or impossible, as is the case in many practical situations; e.g., ablation problems when satellites re-enter the atmosphere, control of welding processes, ovens, or quenching baths, etc.

In this article, we first consider a one-dimensional problem dealing with the identification of the time-dependent heat flux on one side of a one-dimensional linear thermal wall, from temperature and heat flux measurements on the other side. The purpose of this "academic" one-dimensional study is to analyze the influence of the discretization and regularization parameters used in the identification method, and to test its robustness to handle noisy measurements. We then focus on a more realistic situation: our interest is with the identification of the time- and space-dependent heat flux on the boundary of a metal piece during a quenching process. In such a case, temperature measurements are performed inside the material.

Those identification problems, or inverse problems, are ill-posed. "Ill-posed" means that the conditions of Hadamard[1] on existence, uniqueness, and continuity of the solution, with respect to the measured data, are not satisfied. In our problem, this is related to the fact that the input to be restored (boundary heat flux) and the measured output (e.g., temperature inside the material) are connected by a low-pass band filter, so that noise in the system or in the output measure-

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ments destabilizes inverse resolution procedures made to restore the input.

A great deal of attention has been paid to ill-posed problems during the last 30 yr.[2-5] Indeed, they are met in many different fields. For the heat equation, many of the first studies, performed by heat transfer specialists, were analytical.[6-8] Mathematicians, Russian specialists in "functional analysis" in particular, have produced theoretical studies and established existence, uniqueness,[9-11] or stability[12-14] of the solutions. Many other studies are concerned with numerically resolving inverse problems. Most of these studies introduce a regularization technique, which means that they intentionally reduce the admissible input functional space. This is obtained either by choosing the function basis[15-18] or by turning the problem into the minimization of a stabilizing least-square objective function.[11,19-27] Some studies develop very specific approaches, such as space-marching technics.[28]

Our approach is based on the Tikhonov[19,21] regularization technique. A brief description of the method used is given in Sec. II. This section is devoted to the study of the coupled influences of the regularization parameter and the time and space discretizations parameters in a one-dimensional and single-input case. In Sec. III, we shift to a two-dimensional multi-input situation and study the influence of measurements localization. In both cases, time discontinuous input are considered in order to test the method accuracy in the high-frequency domain.

II. One-Dimensional Single-Input Identification Problem

A. Problem Definition and Identification Method

The identification problem is that the unknown input $u(t)$ (to be identified) is the impinging heat flux on one face ($x = 1$) of a one-dimensional linear diffusive wall. On the other side of the wall ($x = 0$), the other input is a flux which is known and equal to zero in the examples given below. The output $y(t)$ is the temperature on this very same face ($x = 0$). The state equation of the temperature $T(x, t)$ is

$$\partial T(x, t)/\partial t = \partial^2 T(x, t)/\partial x^2 \qquad 0 < x < 1 \tag{1}$$

with $x = x_{\dim}/l$ and $t = t_{\dim}a/l^2$, where a and l are the thermal diffusivity and the thickness of the medium, respectively. The initial condition is $T(x, 0) = 0$.

Using a regularization method, we deduce the solution $u^*(t)$ from the measured output $y_m(t)$ by minimization of criterion J, defined on the observation horizon Γ

$$J(u) = \int_0^\Gamma (y - y_m)^2 \, dt + \beta \int_0^\Gamma (u - u_0)^2 \, dt \tag{2}$$

u_0 is a constant, and $y(t)$ is the temperature on face $x = 0$, when the heat flux on face $x = 1$ is $u(t)$. The first term of criterion J handles the information given by the measurements: minimizing this term yields the input $u(t)$ which gives on face $x = 0$ a temperature as close as possible to the measured one. The second part of J is a regularization term, which prevents the minimization of the first term of J from generating instability, using some a priori information on the unknown input. This penalization term ensures the continuity of the inverse operator and the filtering of measurement errors. The relative weight of those two terms is fixed by the choice of the regularization parameter β.

The model used in the numerical procedure to calculate y in terms of u is obtained by a classical spatial discretization (finite difference scheme) of Eq. (1), which becomes

$$\dot{T}(t) = AT(t) + Bu(t) \tag{3}$$

$$y(t) = CT(t) \tag{4}$$

The state vector $T(t) \in R^n$ [the ith component T_i is the temperature at node i, with $x_i = (i\delta x - \delta x/2)$, with $\delta x = 1/n$]. $A(n, n)$, $B(n, 1)$ and $C(1, n)$, are constant matrices easily deduced from the spatial discretization scheme.

Rewriting $J(u)$ with $y(t)$ expressed by Eq. (4) leads to

$$J(u) = \int_0^\Gamma J1 \, dt + \text{const} \tag{5}$$

with

$$J1 = (T^T u) \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} T \\ u \end{pmatrix} + 2(q^T k^T) \begin{pmatrix} T \\ u \end{pmatrix}$$

$$Q = C^T C, \quad R = \beta, \quad q = -C^T y_m, \quad k = -\beta u_0$$

Hence, our identification problem is equivalent to the minimization of a nonhomogeneous quadratic criterion, associated to a linear system. The analytic solution of such a problem is known[29,30] and is given by

$$u^*(T, t) = -R^{-1}[(B^T P)T + B^T g + k] \tag{6}$$

where P is a symmetric matrix of dimension (n, n) solution of a nonstationary Riccati equation, and g is a n-vector solution of a first-order differential equation, with time-dependent coefficients.

B. Test Problem

We will use the following notations:

1) u_V is the input to be identified. Its exact value is used as a reference to check the precision of the identification procedure defined by Eqs. (3–6). To analyze the accuracy of the method in the high-frequency region, $u_V(t)$ is defined for $0 \leq t \leq 1.5$, as: $u_V(t) = -S(t) + 2S(t - 0.5) - 2S(t - 1)$ where $S(t)$ is the step function (c.f. solid line on Fig. 2).

2) y_V is the output generated by u_V.

3) y^* is the solution of Eq. (4) when $u = u^*$.

4) Two errors terms ERu and ERy are defined as

$$ERu = \left\{ (1/M) \sum_{m=1}^{M} [u_V(m\delta t) - u^*(m\delta t)]^2 \right\}^{1/2}$$

$$ERy = \left\{ (1/L) \sum_{l=1}^{L} [y_V(l\delta t) - y^*(l\delta t)]^2 \right\}^{1/2}$$

with $M\delta t = 1.35$ and $L\delta t = \Gamma = 1.5$.

These two terms give rough-averaged information only on the identification performances, but comparing their respective dependance on β is enlightening to understanding the difficulty of such ill-posed problems. Note that the error ERu is not calculated on the whole measurement horizon Γ. Indeed, an error on the input u at the end of the horizon has a negligible effect on $(y - y_m)$. So, at the end of the horizon, when $J(u)$ reaches a minimum, u^* is close to u_0, and not to u_V. The importance of this effect depends mainly on β. It has been shown in Guerrier[25] that the choice of u_0 is not sensitive; a rough estimate of the average order of magnitude of u is good enough. In our example, $u_0 = -\frac{1}{3}$.

The numerical procedure is the following; in a first step, matrix P and vector g are obtained by solving the Riccati equation backward in time from $t = \Gamma$ to $t = 0$. Then, the input $u^*(t)$ is obtained by solving Eqs. (6) and (3) from $t = 0$ to $t = \Gamma$. A fourth-order Runge-Kutta scheme was used, with time step δt for u^* and $\delta t/2$ for P and g.

C. Noiseless Observations

To assess the method ability to identify high-frequency input, we first consider the case of observations with no noise ($y_m = y_V$), and analyze the influence of the parameters β

(regularization parameter) and δt (temporal discretization). The results given in Table 1 and Figs. 1 and 2 for $\delta x = \frac{1}{12}$, lead to the following conclusions:

1) If β gets smaller than some value $\beta_{\min}$, *ERu* increases suddenly and the minimization of $J(u)$ no longer allows the identification of $u(t)$. This minimum value $\beta_{\min}$ decreases when δt decreases (Fig. 1). Let us emphasize that when β is smaller than $\beta_{\min}$, the large increase of *ERu* does not show on *ERy*. Indeed, the difference between u_V and u^* appears chiefly in the high-frequency region, while, due to the low pass-band effect, the contributions of the high frequencies of u on y are very small (Table 1).

2) When β is close to $\beta_{\min}$, Fig. 2 shows that the method gives very satisfactory results, even for high frequencies. Characteristics fluctuations appear when u_V is discontinuous, due to the Gibbs phenomenon.[31]

3) Increasing β entails more filtering of the high frequencies of u_V. This is why *ERu* increases slowly with β (Fig. 1), when $\beta > \beta_{\min}$. A time step δt equals to 0.01/4 (that is $\delta t/(\delta x)^2 \approx 0.4$) is small enough to solve the inverse problem. The minimum error, obtained for $\beta = \beta_{\min}$, is not decreased by use of a smaller time step $\delta t = 0.01/8$, as can be seen in Fig. 1. Further studies[25] have shown that the conclusions are still valid for $\delta x = \frac{1}{8}$, and that the classical stability criterion used in the explicit direct resolution of the diffusive equation, i.e., $\delta t/(\delta x)^2 < 0.5$, can still be used in the numerical resolution of the inverse problem.

Table 1 Influence of β on *ERu* and *ERy* with $n = 12$, $\delta t = 0.01/4$, $\sigma_Y = 0$

β	*ERu*	*ERy*
4×10^{-12}	0.28	0.5×10^{-6}
7×10^{-12}	0.18	0.4×10^{-6}
7×10^{-11}	0.10	0.5×10^{-6}
7×10^{-10}	0.11	10^{-6}
7×10^{-9}	0.12	0.5×10^{-5}
7×10^{-8}	0.14	0.2×10^{-4}
7×10^{-5}	0.24	10^{-3}

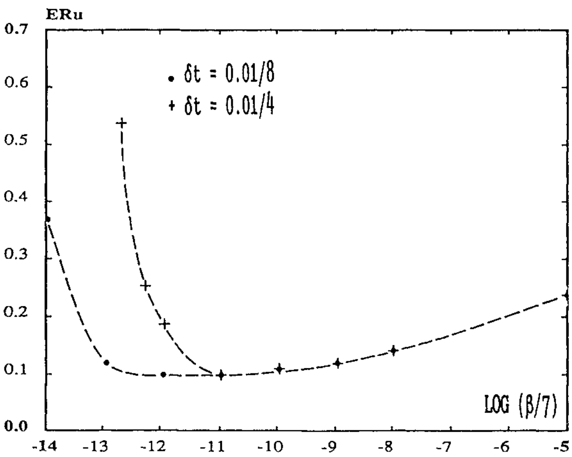

Fig. 1 Influence of δt and β on *ERu* with $n = 12$.

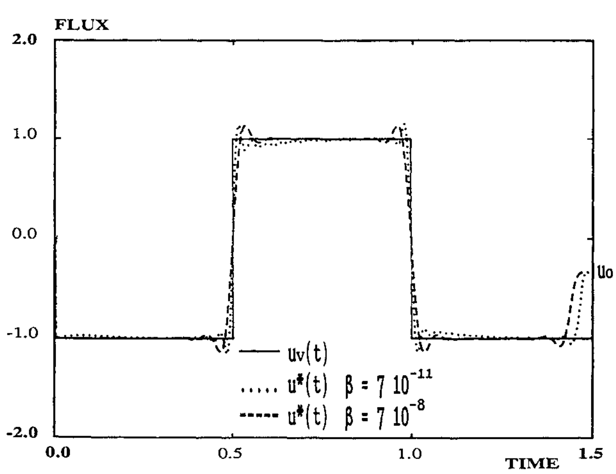

Fig. 2 Comparison of $u_V(t)$ and $u^*(t)$, with $n = 12$, $\delta t = 0.01/4$, $\sigma_Y = 0$.

Table 2 Influence of β on *ERu* and *ERy* with $n = 8$, $\delta t = 0.01/2$, $\sigma_Y = 0.1$

β	*ERu*	*ERy*
7×10^{-6}	1.9	10^{-2}
7×10^{-5}	0.48	0.8×10^{-2}
10^{-4}	0.35	0.8×10^{-2}
4×10^{-4}	0.29	0.8×10^{-2}
10^{-3}	0.33	10^{-2}
7×10^{-3}	0.45	3×10^{-2}
7×10^{-2}	0.74	10^{-1}

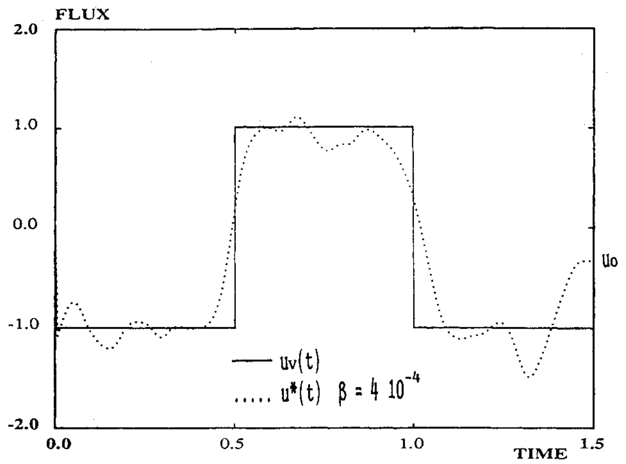

Fig. 3 Comparison of $u_V(t)$ and $u^*(t)$, with $n = 8$, $\delta t = 0.01/2$, $\sigma_Y = 0.1$.

D. Noisy Observations

The purpose of the previous section was to analyze the accuracy of the identification procedure in optimal conditions, i.e., when measurement errors are assumed to be negligible. This section is devoted to the treatment of noisy observations. An additive measurement noise, obtained by simulation of a centered Gaussian white noise, is taken into account. This additive Gaussian noise can simulate errors due to all the measurement equipments. The standard deviation σ_Y is set to 0.1, that is about 21% of the deviation $y_{V\max} - y_{V\min}$. This very small signal-to-noise ratio would, of course, correspond to unusually poor measurements, but the idea is to show that the identification method considered in this article is still robust in such a case. β is used here to handle the high frequencies components introduced in the data by the measurement errors; as a consequence, it is much greater than $\beta_{\min}$. The results in Table 2 indicate that the optimal value of β is about 4×10^{-4} for $\sigma_Y = 0.1$, leading to u^* displayed on Fig. 3. As seen in this figure, the estimation of u_V is still satisfactory in this very noisy case. Note that the variations of *ERu* are not very large for β varying in the range of its optimal value, indicating that the exact value of β is not a very sensitive parameter; $ERu < 0.35$ when $10^{-4} < \beta < 10^{-3}$.

III. Two-Dimensional Multi-Input Identification Problem

Taking advantage of the results obtained in the one-dimensional case, this section deals with a more complex situation, where the unknown boundary is not a time-dependent function, but a time and space one. More precisely, the problem at stake is the identification of the heat flux at the surface of a cylindrical metal piece during a quenching process. Since direct measurements of the flux or temperature on the piece surface would not be reliable, this identification has to be performed from inside temperature measurements.

A. Problem Definition: Direct Simulation

The order of magnitude of the parameters used in this study are those of a quenching process in motor industry. To simulate the quenching process, the following test problem is defined: at $t < 0$, a cylinder is in equilibrium with the outside temperature, T_{ex}. At $t = 0^+$, the cylinder falls into the quenching bath, and is totally immersed at $t_{\dim} = 1$ s. The difference between T_{ex}, the initial temperature, and T_{bath}, the bath temperature, is about 700°C. The exchanges between the cylinder and the bath are driven by boundary conditions of the third kind with $h \approx 2000$ W/m²/C. [In a real process, h would of course change with the different types of heat exchange occurring during the quenching. In this study (devoted to test the method ability to handle discontinuities of the heat flux) it would have been of no use to introduce a variable h.] The state equation is (cylindrical coordinates)

$$\partial T(r, z, t)/\partial t = \partial^2 T/\partial r^2 + (1/r)\, \partial T/\partial r + \partial^2 T/\partial z^2 \qquad (7)$$

with $r = r_{\dim}/\text{radius}$, $z = z_{\dim}/\text{radius}$, height/radius = 12, $t = t_{\dim} a/\text{radius}^2$, and flux = $\text{flux}_{\dim}(\text{radius}/\text{conductivity})/(T_{ex} - T_{\text{bath}})$.

This well-posed direct problem is solved by a classical numerical scheme. Figures 4 and 5 give the temperature profiles on the faces $z = 0$ and $r = 1$. It can easily be seen in Fig. 5 that the two-dimensional effects, at the top and bottom of the cylinder, are not sensitive in the central part. The small slope of the temperature profiles in this zone ($3 < z < 9$) is due to the fall delay. On the contrary, two-dimensional conduction effects create important temperature gradients (≈100°C) in the top and bottom parts of the cylinder, for $0 < z < 3$ and $9 < z < 12$.

The time evolution of the flux $U(t)$ is given in solid line in Figs. 7–11 for different zones: as in the one-dimensional case, $U(t)$ is discontinuous in time, so that this test problem allows us to analyze the method accuracy in the high-frequency domain.

B. Output: Measurements

Temperature measurements in the two-dimensional sensitive zone are obviously needed if we want to get some information on the two-dimensional effects. In order to study the sensitivity of the identification to the thermocouples localization, two situations are analyzed. For both of them, temperatures are supposed to be measured in nine points inside the cylinder (Table 3) (c.f. Fig. 6 for test 1).

The only difference between the two tests is the distance of the thermocouples 1, 2, 3 and 4, 5, 6 from the bottom and top, respectively. This distance is 0.5 in test one and 0.23 in test 2. These distances are to be compared with the total height of the cylinder which is $z = 12$. Only a few thermocouples are used in the central part, since we know that more data would be of no use in this test problem.

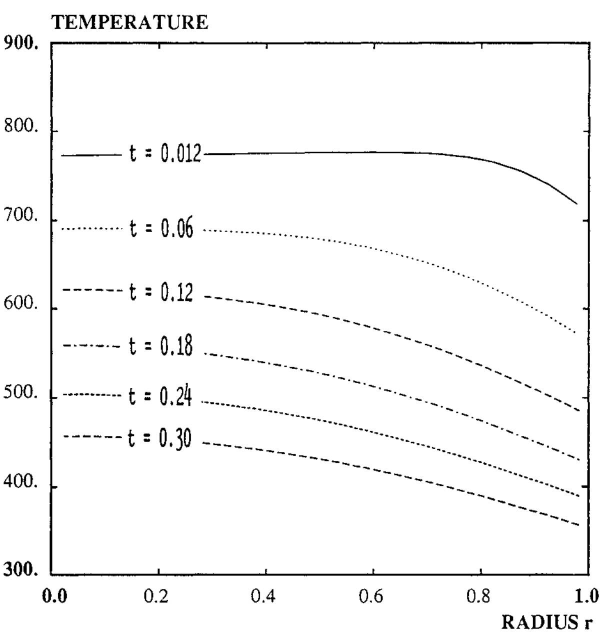

Fig. 4 $T(0 \le r \le 1, z = 0, t)$: **Temperature gradient on face** $z = 0$.

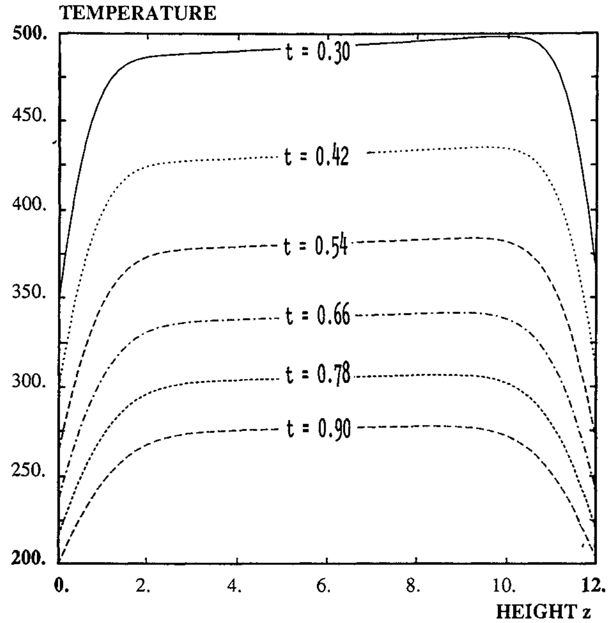

Fig. 5 $T(r = 1, 0 \le z \le 12, t)$: **Temperature gradient on face** $r = 1$.

Table 3 Thermocouples localization

Output	r	Test 1: z	Test 2: z
1	0	0.5	0.23
2	0.48	0.5	0.23
3	0.83	0.5	0.23
4	0	11.5	11.77
5	0.48	11.5	11.77
6	0.83	11.5	11.77
7	0.83	1.53	1.53
8	0.83	6	6
9	0.83	10.5	10.5

C. Identification Method

From now on, the cylinder initial temperature and the temperatures of the nine thermocouples is the only information we have to perform the heat flux identification. The purpose of the study is to analyze the method ability to restore both the space distribution and the time evolution of the unknown heat flux.

The identification method will not be detailed here since it is similar to the one-dimensional case. The original heat Eq. (7) is discretized, in order to get a set of ordinary differential equations. We use a geometrical progression to generate the grid in the r and z directions. This irregular discretization scheme allows us to reduce the dimension of the state vector on one hand, and to keep a precise grid near the boundaries on the other hand. Seven and fifteen nodes are defined in the r ($0 \le r \le 1$) and z ($0 \le z \le 12$) directions, respectively: $n_r = 7$, $n_z = 15$, $n = n_r n_z$. In both directions, the space interval close to the boundary is about $\frac{1}{12}$, as in the one-dimensional case. The boundary condition vector $\boldsymbol{Ucl}(t)$, discretized cor-

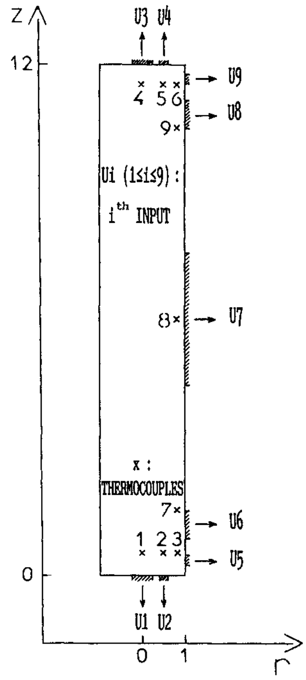

Fig. 6 Test 1: Identified input and thermocouples localization.

respondingly, has $2n_r + n_z$ components. The discretized direct model used in the identification procedure is

$$\dot{T}(t) = AT(t) + BUcl(t) \tag{8}$$

$$Y(t) = CT(t) \tag{9}$$

Because of the large aspect ratio of the piece on one hand, and the small number of temperature measurements on the other hand, identification of all the discretized components of $\boldsymbol{Ucl}(t)$ would have no meaning. Therefore, the input vector to be identified, $U(t)$, is limited to nine components. Figure 6 gives the nine selected components of $\boldsymbol{Ucl}(t)$ to be identified, and the corresponding meshes interval of the direct model [Eqs. (8) and (9)]. In the same way, as for the thermocouples, the choice of the input localization is made to test the method ability to restore the two-dimensional effects.

As in the one-dimensional case, $\boldsymbol{U}^*(t)$ is deduced from the measured output $\boldsymbol{Ym}(t)$ by minimizing the following criterion J, defined on the observation horizon Γ ($\Gamma = 0.96$):

$$J(U) = \int_0^\Gamma (Y - Ym)^T(Y - Ym)\,\mathrm{d}t + \int_0^\Gamma (U - U_0)^T MU(U - U_0)\,\mathrm{d}t$$

where a 9×9 regularization matrix MU has been introduced.

To compute U, the whole set of discretized $(2nr + nz)$ boundary conditions is needed. Therefore, the unidentified boundary conditions must be expressed in function of the identified ones. For the "corners meshes" ($n_r = 7$, $n_z = 1$ or 15) the heat flux for the horizontal face is assumed to be equal to the vertical one. All other unidentified boundary conditions are obtained by linear interpolation of the two closest identified one. The following parameters are used: $\delta t = 0.0024$, $MU = 10^{-4}Id$, $U_{0j} = 100$.

D. Results

Figures 7–11 on one hand, and Figs. 12 and 13 on the other hand give the identification results for test 1 and test 2, respectively. Comparison of these results leads to the following conclusions:

1) On the vertical face (input 5–9), the method succeeds in restoring both the time delay due to the fall in the quenching bath, and the time evolution of the heat flux. For example, for test one, Figs. 9–11 give the identified heat flux for input 5, 7, and 8.

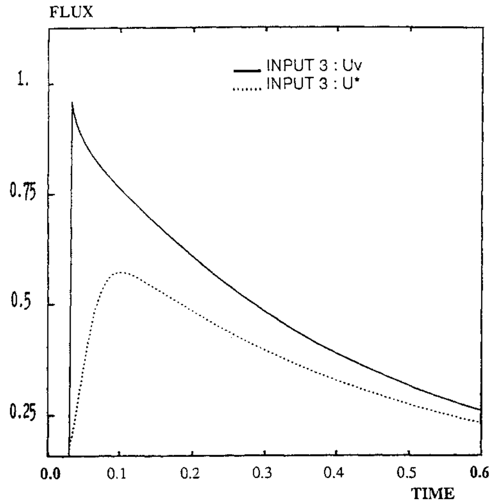

Fig. 7 Test 1: Comparison of $U3_V$ and $U3^*$ ($z = 12$).

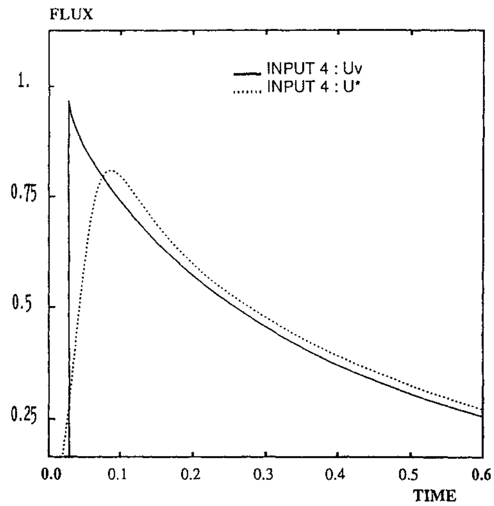

Fig. 8 Test 1: Comparison of $U4_V$ and $U4^*$ ($z = 12$).

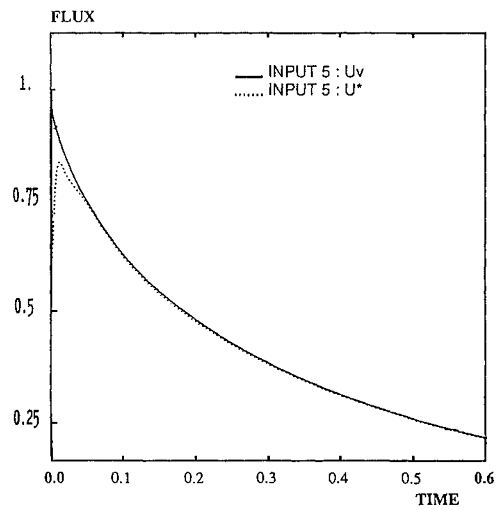

Fig. 9 Test 1: Comparison of $U5_V$ and $U5^*$ ($r = 1$).

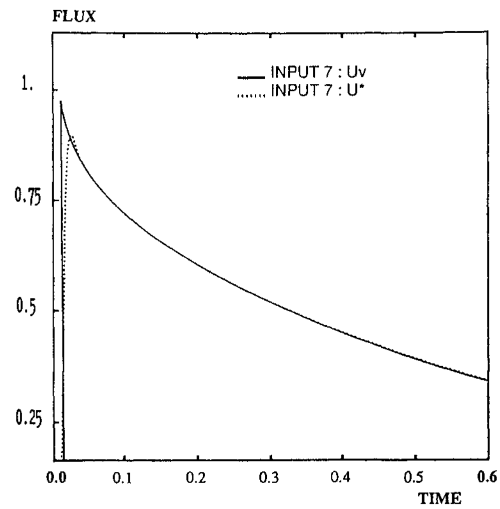

Fig. 10 Test 1: Comparison of $U7_V$ and $U7^*$ (r = 1).

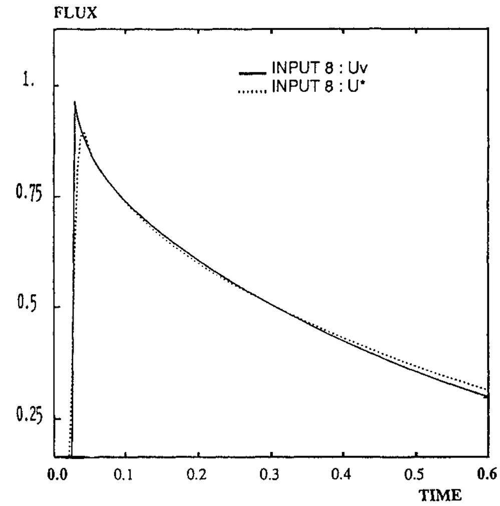

Fig. 11 Test 1: Comparison of $U8_V$ and $U8^*$ (r = 1).

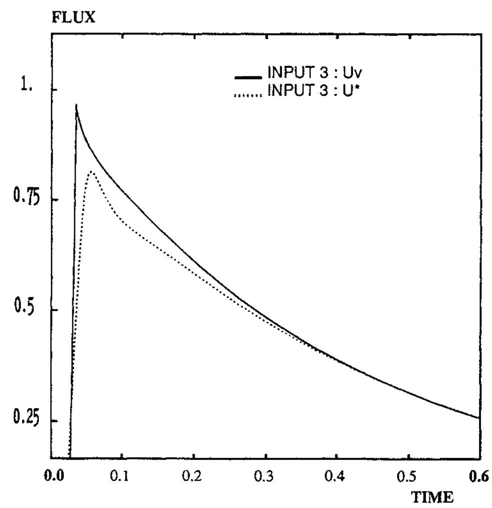

Fig. 12 Test 2: Comparison of $U3_V$ and $U3^*$ (z = 12).

2) Comparison of the identified heat fluxes on the horizontal faces (input 1–4), for the two set of measurements, shows the difficulties of this multivariable identification. For test 1, Figs. 7 and 8 give the identified input U3 and U4 (top of the cylinder). In spite of their localization in the two-dimensional sensitive zone (0.5 from the top), thermocouples 4, 5, 6 are too far away from the top surface to allow a good

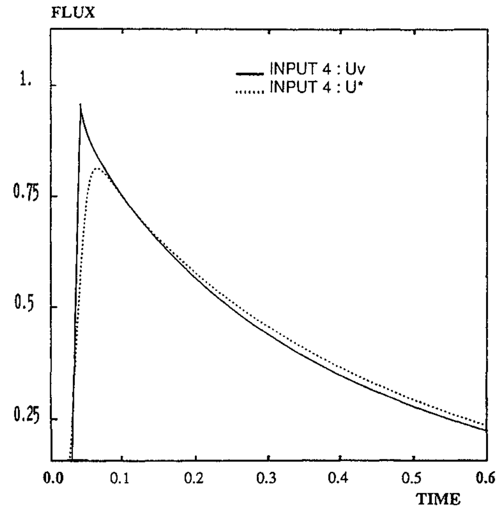

Fig. 13 Test 2: Comparison of $U4_V$ and $U4^*$ (z = 12).

spatial discrimination. Two-dimensional diffusion effects are strong enough for U3 to be underestimated and U4 to be overestimated, while the time-averaged error ERy on each of the nine output seems very satisfactory. Indeed, $ERy/[Y(t = 0) - Y(t = \Gamma)]$ is no more than 10^{-3}. The results obtained for the bottom of the cylinder (inputs 1 and 2) are very similar.

It is only when considering the second set of measurements, with thermocouples closest to the top and bottom of the cylinder (0.23 instead of 0.5), that the two-dimensional effects are correctly restored, as shown in Figs. 12 and 13.

This example highlights the strong ill-posed nature of this multi-input problem; the information brought by the measurements is not selective enough, so that the criterion J does not show a sharp minimum. It means that multi-input identification in a real quenching process would need temperature meaurements in the quenched piece very close to its surface, if accurate spatial discrimination is wanted.

IV. Conclusions

The purpose of the present work has been to analyze a regularization procedure in a one-dimensional and two-dimensional identification problem.

In the one-dimensional single-input case, we have analyzed the role of the discretization and regularization parameters. In the case of noiseless observations, we have emphasized that β must be greater than some value β_{min} in order to solve the discretized inverse problem numerically. When β is close to β_{min}, the method was shown to restore the unknown input with a very good accuracy, even in the high frequency region: discontinuities in the flux are well-restored. With a proper choice of β, the method is still robust to solve very small signal-to-noise ratio, when the noise standard deviation is more than 20% of the range of variation of the measured output. Regarding the discretization problem, it has been shown that the usual stability criterion used to solve the direct explicit problem, i.e., $\delta t/(\delta x)^2 < 0.5$, is still valid for the resolution of the inverse problem with the method studied in this article.

Taking advantage of the results obtained in the one-dimensional case, we have analyzed the sensitivity of the identification to the thermocouples localization in a two-dimensional multi-input case. As long as the sensors are close enough to the boundary, the method was shown to restore both temporal and spatial evolution of the heat flux, with good accuracy.

References

[1]Hadamard, J., *Lectures on the Cauchy Problem in Linear Partial Differential Equations*, Yale Univ. Press, New Haven, CT, 1923.

[2]Bertero, M., "Regularization Methods for Linear Inverse Prob-

lems," *Inverse Problems, Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 1986, pp. 52–112.

[3]Tarantola, A., *Inverse Problem Theory, Methods for Data Fitting and Model Parameter Estimation*, Elsevier, Amsterdam, The Netherlands, 1987.

[4]Engl, H. W., and Groetsch, C. W. (eds.), "Inverse and Ill-Posed Problems," *Notes and Reports in Mathematics in Science and Engineering*, Vol. 4, Academic Press, Boston, MA, 1987.

[5]Hensel, E., *Inverse Theory and Applications for Engineers*, Prentice-Hall, Englewood Cliffs, NJ, 1990.

[6]Burggraf, O. R., "An Exact Solution of the Inverse Problem in Heat Conduction Theory and Applications," *Journal of Heat Transfer*, Vol. 3, Aug. 1964, pp. 373–382.

[7]Sparrow, E. M., Haji-Sheikh, A., and Lundgren, T. S., "The Inverse Problem in Transient Heat Conduction," *Journal of Applied Mechanics*, Vol. 3, Sept. 1964, pp. 369–375.

[8]Imber, M., "Non-Linear Transfer in Planar Solids, Direct and Inverse Applications," *AIAA Journal*, Vol. 17, No. 2, 1978, pp. 204–212.

[9]Lazuchenkov, N. M., and Skmukin, A. A., "Inverse Boundary-Value Problem of Heat Conduction for a Two-Dimensional Domain," *Journal of Engineering Physics*, 1981, pp. 223–228 (translated from *Inzhenerno-Fizicheskii Zhurnal*, Russian, Vol. 40, No. 2, 1981, pp. 352–358).

[10]Romanovskii, M. R., "Identifiability in the Large of a Linear Heat-Conduction Equation Subject to Cauchy Boundary Conditions," *Journal of Engineering Physics*, 1983, pp. 320–326 (translated from *Inzhenerno-Fizicheskii Zhurnal*, Russian, Vol. 44, No. 3, 1983, pp. 457–465).

[11]Bulychev, E. V., and Glasko, V. B., "Uniqueness in Certain Inverse Problems of the Theory of Heat Conduction," *Journal of Engineering Physics*, 1984, pp. 940–943 (translated from *Inzhenerno-Fizicheskii Zhurnal*, Russian, Vol. 45, No. 2, 1983, pp. 305–309).

[12]Elden, L., "Modified Equations for Approximating the Solution of a Cauchy Problem for the Heat Equation," *Inverse and Ill-Posed Problems*, edited by H. W. Engl, and C. W. Groetsch, *Notes and Reports in Mathematics in Science and Engineering*, Vol. 4, Academic Press, Boston, MA, 1987.

[13]Knabner, P., and Vessella, S., "Stability Estimates for Ill-Posed Cauchy Problems for Parabolic Equations," *Inverse and Ill-Posed Problems*, edited by H. W. Engl, and C. W. Groetsch, *Notes and Reports in Mathematics in Science and Engineering*, Vol. 4, Academic Press, Boston, MA, 1987.

[14]Borukhov, V. T., and Kolesnikov, P. M., "Method of Inverse Dynamic Systems and Its Application for Recovering Internal Heat Sources," *International Journal of Heat and Mass Transfer*, Vol. 31, No. 8, 1988, pp. 1549–1556.

[15]Takahashi, R., and Arakawa, A., "Real Time Application of Simultaneous Estimation Theory for Parameters and State to One-Dimensional Heat Conduction," *Nuclear Engineering and Design*, North Holland, Amsterdam, The Netherlands, 1984, pp. 437–449.

[16]Hsieh, C. K., and Kassab, A. J., "A General Method for the Solution of Inverse Heat Conduction Problems with Partially Unknown System Geometries," *International Journal of Heat and Mass Transfer*, Vol. 29, No. 1, 1986, pp. 47–58.

[17]Tsoi, P. V., "A Method for Calculating Direct and Inverse Problems of Unsteady-State Heat Transfer," International Journal of Heat and Mass Transfer, Vol. 31, No. 3, 1988, pp. 497–504.

[18]Flach, G. P., and Ozisik, M. N., "Inverse Heat Conduction Problem of Periodically Contacting Surfaces," *Journal of Heat Transfer*, Vol. 110, Nov. 1988, pp. 821–829.

[19]Tikhonov, A. N., "Solution of Incorrectly Formulated Problems and the Regularization Method—Regularization of Incorrectly Posed Problems," *Soviet Math. Dokladi*, Vol. 4, 1963, pp. 1035–1038, 1624–1627.

[20]Lavrentiev, M. M., *Some Improperly Posed Problems of Mathematical Physics*, Springer-Verlag, Berlin, 1967.

[21]Tikhonov, A. N., and Arsenin, V. Y., *Solutions of Ill-Posed Problems*, translation edited by F. John, Winston/Wiley, Washington, DC, 1977.

[22]Alifanov, O. M., and Mikhailov, V. V., "Solution of the Inverse Boundaryvalue Problem of Heat Conduction in an Overdefined Formulation," *Journal of Engineering Physics*, 1984, pp. 1270–1274 (translated from *Inzhenerno-Fizicheskii Zhurnal*, Russian, Vol. 45, No. 5, 1983, pp. 776–781).

[23]Busby, H. R., and Trujillo, D. M., "Numerical Solution to a Two-Dimensional Inverse Heat Conduction Problem," *International Journal for Numerical Methods in Engineering*, Vol. 21, 1985, pp. 849, 859.

[24]Benard, C., Guerrier, B., and Rosset, M. M., "Identification or Control of the Boundary Inputs of a Linear Thermal System with No Intrusive Measurement, in the Presence of Internal Heat Source Disturbances," *Proceedings of the 8th International Heat Transfer Conference*, Vol. 2, San Francisco, CA, 1986, pp. 665–670.

[25]Guerrier, B., "Relation Entrées-Sorties dans des Systèmes Diffusifs: Réduction du Modèle d'État—Identification de Conditions aux Limites," *Thèse de Doctorat d'Etat*, Université Paris VI, Paris, France, June 1989.

[26]Afshari, A., Benard, C., Duhamel, C., and Guerrier, B., "On-Line Identification of the State of the Surface of a Material Undergoing Thermal Processing," *Proceedings of the 5ème Symposium IFAC "Commande des Systèmes à Paramètres Distribués,"* Perpignan, France, June 1989, pp. 459–463.

[27]Scott, E. P., and Beck, J. V., "Analysis of Order of the Sequential Regularization Solutions of Inverse Heat Conduction Problems," *Journal of Heat Transfer*, Vol. 111, May 1989, pp. 218–224.

[28]Hensel, E., and Hills, R. G., "An Initial Value Approach to the Inverse Heat Conduction Problem," *ASME Transactions*, Vol. 108, May 1986, pp. 248–256.

[29]Kalman, R. E., "Contributions to the Theory of Optimal Control," *Boletin de la Sociedad Matematica Mexicana*, Vol. 5, No. 1, 1960, pp. 102–119.

[30]Athans, M., and Falb, P., *Optimal Control*, McGraw-Hill, New York, 1966.

[31]Papoulis, A., *The Fourier Integral and its Applications*, McGraw-Hill, New York, 1962.